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Note

A Note on Multiply Transitive Permutation Groups

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In [1] Hall determined all 4-ply transitive permutation groups whose stabilizer of 4 points is of odd order. See also Nagao [2].

In this short note we give a theorem like Hall and Nagao for any odd primes but for permutation groups with relatively high multiple transitivity.

THEOREM. *Let p be an odd prime. Let G be a p^2-ply transitive permutation group on a set $\Omega = \{1,..., n\}$. Assume that a Sylow p-subgroup of a stabilizer of p^2 points $1,..., p^2$ in G is not the identity and semiregular on $\Omega - \{1,..., p^2\}$. Then we obtain that $n = p^2 + p$, and $G = S^\Omega$ or A^Ω (the symmetric and alternating groups on Ω).*

COROLLARY. *Let p be an odd prime. Let G be a t-ply transitive permutation group on a finite set Ω such that the order of the stabilizer of t points in G is not divisible by p and that G does not contain A^Ω. Then we obtain that $t < p^2 + p$.*

Proof of Corollary. Let $t \geq p^2 + p$. If $n - t + i$ (with $1 \leq i \leq p$) is divisible by p, then the stabilizer of $t - p^2 - i$ points in G satisfies the assumption of the theorem. Thus, we obtain the assertion.

Proof of Theorem. We use the following lemma.

LEMMA. *Let G satisfy the assumption of the Theorem. Let H be a p-subgroup of G and let Δ be a subset of Ω fixed as a whole by H with $|\Delta| = p^2$. Then H fixes at least $p - 1$ elements of order p of the stabilizer of Δ in G.*

Proof. Proof is obvious from the fact that H normalizes a Sylow p-subgroup of the stabilizer of Δ in G. Q.E.D.

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In the following we assume that G does not contain A^{Ω}. Therefore, we may assume that n is not so small. There exists an element a of order p of G such that $|I(a)| = p^2$, where $I(a)$ denotes the set of the fixed points of Ω by a. Let us take p points, say $\alpha_1, ..., \alpha_p$, from $I(a)$. Let $(\alpha_{p+1}, ..., \alpha_{2p})$ be a p-cycle of a. Then, by Lemma, there exists an element b of order p of the stabilizer $(I(a) - \{\alpha_1, ..., \alpha_p\}) \cup \{\alpha_{p+1}, ..., \alpha_{2p}\}$ in G such that b is commutative with a. Then the subgroup $\langle a, b\rangle$ generated by a and b contains $p^2 - p$ orbits of length 1 and at most $p + 1$ orbits of length p and other orbits are of length p^2. Now let $\{\beta_1, ..., \beta_{p^2}\}$ be an orbit of $\langle a, b\rangle$ of length p^2. Also by Lemma, there exists an element c of order p of the stabilizer of $\{\beta_1, ..., \beta_{p^2}\}$ in G such that c commutes with $\langle a, b\rangle$. Now c contains two p-cycles on $\{\alpha_1, ..., \alpha_p\}$ and $\{\alpha_{p+1}, ..., \alpha_{2p}\}$. Let $(\alpha_{2p+1}, ..., \alpha_{3p})$ be another p-cycle of c on $I(a)$. Then, also by Lemma, there exists an element d of order p of the stabilizer in G of $I(a) \cup \{\alpha_{p+1}, ..., \alpha_{2p}\} - \{\alpha_{2p+1}, ..., \alpha_{3p}\}$ such that d commutes with $\langle a, b, c\rangle$. Since d is commutative with $\langle a, b\rangle$, there exists a nonidentity element e of $\langle d, a, b\rangle$ such that e acts trivially on $\{\beta_1, ..., \beta_{p^2}\}$. Since e has the nontrivial (because $p \geqslant 3$) fixed subset $I(a) \cap I(b) \cap I(d)$, we obtain that $|I(e)| > p^2$. But this is a contradiction, and the proof is completed.

References

1. M. Hall, On a theorem of Jordan, *Pacific J. Math.* **4** (1954), 219–226.
2. H. Nagao, On multiply transitive groups V, *J. Algebra* **9** (1968), 240–248.

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